Българска академия на науките • Bulgarian Academy of Sciences
Аерокосмически изследвания в България • 13 • Aerospace Research in Bulgaria
София • 1997 • Sofia

On the perturbative gravity and quantum gravity theory on a curved background. II. Application of covariant derivatives and third rank tensors*

(Revised version)

Bogdan Dimitrov

High Energy Astrophysics Section,
Space Research Institute, Bulgarian Academy of Sciences

Preliminary notes

In this paper all expressions, derived in the preceeding paper [1] will be simplified and written in another way so that some important physical conclusions can be made. This will be achieved by applying formulae (I.12) and defining a covariant derivative of the tensor field $h_{\mu\nu}$ again in respect to the background symmetric Levi - Civita connection $\Gamma_{\mu\nu}^{(0)\alpha}$:

(1) $$h_{\mu\nu|\alpha} = \partial_{\alpha} h_{\mu\nu} + \Gamma_{\mu\alpha}^{(0)r} h_{r\nu} + \Gamma_{\alpha\nu}^{(0)r} h_{\mu r} \neq 0.$$

The definition is in accordance with the 'minus' sign convention in the expression (I.10) for $\Gamma_{\mu\nu}^{(0)\alpha}$ and the covariant derivative $h_{\mu\nu|\alpha}$ is different from zero because the tensor field $h_{\mu\nu}$ has entirely different properties from the background field $g_{\mu\nu}^{(0)}$.

* Work supported in part by the National Science Foundation in Bulgaria under grant No NIP-753.

Throughout the whole paper the Roman letter I will denote formulaes, used in the previous paper I.

Third-rank tensor fluctuation connection

In order to define this connection, we shall use expression (1) to replace all partial derivatives with their covariant ones in the expression (I.14) for $H^{(1)\alpha}_{\mu\nu}$. Remembering also the basic assumption in paper I about the Riemannian background metric, which means that $g^{(0)}_{\mu\nu|\alpha} \equiv 0$, we derive the following formulae for $H^{(1)\alpha}_{\mu\nu}$:

$$H^{(1)\alpha}_{\mu\nu} = -\tfrac{1}{2} g^{(0)\alpha s}\left(h_{\nu s|\mu} + h_{\mu s|\nu} - h_{\mu\nu|s}\right). \tag{2}$$

An important conclusion can be made from (2) : the quantity $H^{(1)\alpha}_{\mu\nu}$ is a tensor, unlike the background connection $\Gamma^{(0)\alpha}_{\mu\nu}$ and it shall be called first fluctuation connection. Formulae (2) was derived also in [2], but there it was not pointed out that it is valid only under the assumption about the Riemannian nature of the background geometry.

By applying again formulae (2), we can rewrite the expression (I.15) for $H^{(2)\alpha}_{\mu\nu}$ in the following way:

$$H^{(2)\alpha}_{\mu\nu} \equiv P^{(2)\alpha}_{\mu\nu} - h^{\alpha s} h_{rs} \Gamma^{(0)r}_{\mu\nu} \tag{3}$$

where $P^{(2)\alpha}_{\mu\nu}$ is the tensor quantity:

$$P^{(2)\alpha}_{\mu\nu} \equiv \tfrac{1}{2} h^{\alpha s}\left(h_{\nu s|\mu} + h_{\mu s|\nu} - h_{\mu\nu|s}\right). \tag{4}$$

Unfortunately, $H^{(2)\alpha}_{\mu\nu}$ is not a tensor due to the presence of the term $h^{\alpha s} h_{rs} \Gamma^{(0)r}_{\mu\nu}$, but it is important to mention that a tensor $P^{(2)\alpha}_{\mu\nu}$ has been singled out.

First-and second-order Riemannian tensor and covariant derivatives of third-rank tensors

In order to write down more concisely the expressions (I.19) - (I.22) for $R^{(1)\alpha}_{\mu\alpha\nu}$ and $R^{(2)\alpha}_{\mu\alpha\nu}$ let us define the covariant derivative of a third-rank tensor :

$$(5)\qquad H^{(1)\beta}_{\mu\nu|\alpha} \equiv H^{(1)\beta}_{\mu\nu,\alpha} - H^{(1)\rho}_{\mu\nu}\Gamma^{(0)\beta}_{\rho\alpha} + H^{(1)\beta}_{\rho\nu}\Gamma^{(0)\rho}_{\mu\alpha} + H^{(1)\beta}_{\mu\rho}\Gamma^{(0)\rho}_{\mu\alpha}$$

with respect to $\Gamma^{(0)\beta}_{\mu\nu}$. The above definition is consistent with the definition of a covariant derivative of n-th rank tensor, given in [3]. By means of (5) the expression (I.19) for $R^{(1)\beta}_{\mu\alpha\nu}$ acquires the form:

$$(6)\qquad R^{(1)\beta}_{\mu\alpha\nu} \equiv H^{(1)\beta}_{\mu\nu|\alpha} - H^{(1)\beta}_{\mu\alpha|\nu}\,.$$

If we apply the formulaes (3) and (5) to the expression (I.20) for $R^{(2)\beta}_{\mu\alpha\nu}$, we will obtain:

$$(7)\qquad R^{(2)\beta}_{\mu\alpha\nu} \equiv P^{(2)\beta}_{\mu\alpha\nu} - h^{\beta s}h_{rs}R^{(0)r}_{\mu\alpha\nu} + h^{\rho s}h_{rs}\left(\Gamma^{(0)r}_{\mu\nu}\Gamma^{(0)\beta}_{\rho\alpha} - \Gamma^{(0)r}_{\mu\alpha}\Gamma^{(0)\beta}_{\rho\nu}\right)$$
$$+\left\{\left(h^{\beta s}h_{rs}\right)_{,\nu}\Gamma^{(0)r}_{\mu\alpha} - \left(h^{\beta s}h_{rs}\right)_{,\alpha}\Gamma^{(0)r}_{\mu\nu}\right\} + S^{(2)\beta}_{\mu\alpha\nu}$$

where $S^{(2)\beta}_{\mu\alpha\nu}$ is given by the formulaes (I.21) and (I. 22) and $P^{(2)\beta}_{\mu\alpha\nu}$ is the tensor quantity:

$$(8)\qquad P^{(2)\beta}_{\mu\alpha\nu} \equiv P^{(2)\beta}_{\mu\nu|\alpha} - P^{(2)\beta}_{\mu\alpha|\nu}\Gamma^{(0)\beta}_{\rho\alpha} + H^{(1)\rho}_{\mu\alpha}H^{(1)\beta}_{\rho\nu} - H^{(1)\rho}_{\mu\nu}\Gamma^{(0)\beta}_{\rho\alpha}\,.$$

It may be verified that both sides of the expression:

$$(9)\qquad H^{(1)\alpha}_{\mu\nu} \equiv -\tfrac{1}{2}g^{(0)\alpha s}\left(h_{\nu s|\mu} + h_{\mu s|\nu} - h_{\mu\nu|s}\right)$$

can be multiplied by $g^{(0)}_{\alpha s}$ and therefore:

$$(10)\qquad h_{\nu s|\mu} + h_{\mu s|\nu} - h_{\mu\nu|s} \equiv -2g^{(0)}_{\alpha s}H^{(1)\alpha}_{\mu\nu}\,.$$

By means of (10) we obtain:

$$(11)\qquad P^{(2)\beta}_{\mu\nu} \equiv \tfrac{1}{2}h^{\beta s}\left(h_{\nu s|\mu} + h_{\mu s|\nu} - h_{\mu\nu|s}\right) \equiv -g^{(0)}_{\delta s}h^{\beta s}H^{(1)\delta}_{\mu\nu} \equiv -g^{(0)\beta q}h_{\gamma q}H^{(1)\gamma}_{\mu\nu}\,.$$

Also, the Leibniz rule for a covariant derivative of a tensor product is valid:

$$(12)\qquad P^{(2)\beta}_{\mu\nu|\alpha} \equiv -g^{(0)\beta q}h_{\gamma q|\alpha}H^{(1)\gamma}_{\mu\nu} - g^{(0)\beta q}h_{\gamma q}H^{(1)\gamma}_{\mu\nu|\alpha}\,.$$

The covariant derivative $h_{\gamma q|\alpha}$ can easily be expressed through the tensors

$H^{(1)}$ by means of a cyclic change of indices, then summing up the three equations (10) and finally equating the obtained result from (10). The final result is:

$$h_{\gamma q|\alpha} = -\left[g^{(0)}_{pq} H^{(1)p}_{\gamma\alpha} + g^{(0)}_{p\gamma} H^{(1)p}_{q\alpha}\right]. \tag{13}$$

Formulaes (6) - (13) can be used to find a modified expression for $P^{(2)\beta}_{\mu\alpha\nu}$:

$$P^{(2)\beta}_{\mu\alpha\nu} \equiv -h^{\beta}_{\gamma} R^{(1)\gamma}_{\mu\alpha\nu} + g^{(0)\beta q} g^{(0)}_{p\gamma}\left(H^{(1)\gamma}_{\mu\nu} H^{(1)p}_{q\alpha} - H^{(1)\gamma}_{\mu\alpha} H^{(1)p}_{q\nu}\right). \tag{14}$$

It is important also to see that the only terms in the expression (7) for $R^{(2)\beta}_{\mu\alpha\nu}$, which contain partial derivatives, are

$$\left(h^{\beta s} h_{rs}\right)_{,\nu} \Gamma^{(0)r}_{\mu\alpha} \left(h^{\beta s} h_{rs}\right)_{,\alpha} \Gamma^{(0)r}_{\mu\nu} \tag{15}$$

and also the term:

$$F^{(2)\beta}_{\mu\alpha\nu} \equiv \left(h^{st} h_{st}\right)_{,\alpha} \Gamma^{(0)\beta}_{\mu\nu} - \left(h^{st} h_{st}\right)_{,\nu} \Gamma^{(0)\beta}_{\mu\alpha} \tag{16}$$

which is a part of the tensor $S^{(2)\beta}_{\mu\alpha\nu}$ (I.21). However, since it is inconvenient to deal both with partial and covariant derivatives of one and the same tensor field $h_{\mu\nu}$, we will use (1) and also the formulae:

$$h^{\beta s}_{|\alpha} \equiv h^{\beta s}_{,\alpha} - h^{ps}\Gamma^{(0)\beta}_{p\alpha} - h^{\beta p}\Gamma^{(0)s}_{p\alpha} \tag{17}$$

to express all partial derivatives through their covariant ones. Afterwards, all covariant derivatives of $h_{\mu\nu}$ are expressed through the third-rank tensor $H^{(1)\alpha}_{\mu\nu}$ by means of (13) and the following expressions are obtained for (15) and (16):

$$\begin{aligned}
&\left(h^{\beta s} h_{rs}\right)_{,\nu} \Gamma^{(0)r}_{\mu\alpha} - \left(h^{\beta s} h_{rs}\right)_{,\alpha} \Gamma^{(0)r}_{\mu\nu} \equiv W^{\beta}_{1\mu\alpha\nu} + W^{\beta}_{2\mu\alpha\nu} + W^{\beta}_{3\mu\alpha\nu} + W^{\beta}_{4\mu\alpha\nu} \\
&- h^{\beta s} h_{ps} R^{(0)p}_{\mu\alpha\nu} + h^{\beta s} h_{ps}\left(\Gamma^{(0)p}_{\mu\nu,\alpha} - \Gamma^{(0)p}_{\mu\alpha,\nu}\right) - h^{p}_{\gamma} h^{\gamma}_{r}\left(\Gamma^{(0)r}_{\mu\nu}\Gamma^{(0)\beta}_{p\alpha} - \Gamma^{(0)r}_{\mu\alpha}\Gamma^{(0)\beta}_{p\nu}\right),
\end{aligned} \tag{18}$$

$$\begin{aligned}
F^{(2)\beta}_{\mu\alpha\nu} &\equiv \left(h^{st} h_{st}\right)_{,\alpha} \Gamma^{(0)\beta}_{\mu\nu} - \left(h^{st} h_{st}\right)_{,\nu} \Gamma^{(0)\beta}_{\mu\alpha} \\
&\equiv \left(h^{st} h_{st}\right)_{|\alpha} \Gamma^{(0)\beta}_{\mu\nu} - \left(h^{st} h_{st}\right)_{|\nu} \Gamma^{(0)\beta}_{\mu\alpha\nu} \equiv 4 V^{\beta}_{\mu\alpha\nu}
\end{aligned} \tag{19}$$

where $V^{\beta}_{\mu\alpha\nu}$ is the tensor quantity;

$$V^{\beta}_{\mu\alpha\nu} \equiv h^{\gamma}_{\rho}\left(\Gamma^{(0)\beta}_{\mu\alpha} H^{(1)\rho}_{\gamma\nu} - \Gamma^{(0)\beta}_{\mu\nu} H^{(1)\rho}_{\gamma\alpha}\right) \tag{20}$$

and $W^{\beta}_{1\mu\alpha\nu} \div W^{\beta}_{4\mu\alpha\nu}$ are similar tensor quantities:

$$W^{\beta}_{1\mu\alpha\nu} \equiv h^{\beta}_{\rho}\left(\Gamma^{(0)r}_{\mu\nu} H^{(1)\rho}_{r\alpha} - \Gamma^{(0)r}_{\mu\alpha} H^{(1)\rho}_{r\nu}\right), \tag{21}$$

$$W^{\beta}_{2\mu\alpha\nu} \equiv h^{\beta s} g^{(0)}_{\rho r}\left(\Gamma^{(0)r}_{\mu\nu} H^{(1)\rho}_{s\alpha} - \Gamma^{(0)r}_{\mu\alpha} H^{(1)\rho}_{s\nu}\right), \tag{22}$$

$$W^{\beta}_{3\mu\alpha\nu} \equiv g^{(0)\beta q} h_{r\rho}\left(\Gamma^{(0)r}_{\mu\nu} H^{(1)\rho}_{q\alpha} - \Gamma^{(0)r}_{\mu\alpha} H^{(1)\rho}_{q\nu}\right) \tag{23}$$

$$W^{\beta}_{4\mu\alpha\nu} \equiv h^{q}_{r}\left(\Gamma^{(0)r}_{\mu\nu} H^{(1)\beta}_{q\alpha} - \Gamma^{(0)r}_{\mu\alpha} H^{\beta}_{q\nu}\right). \tag{24}$$

Note that expression (19) is invariant under replacement of all partial derivatives with their covariant ones. This property is not valid for expression (18), which is similar in structure to (19), but its tensor indices have a different displacement.

We have obtained all the necessary formulaes for the derivation of $R^{(2)\beta}_{\mu\alpha\nu}$. Using formulaes (7), (8), (I.21), (I.22), (14), (18) - (24), we can obtain the following modified expression for the second-order Riemann tensor $R^{(2)\beta}_{\mu\alpha\nu}$:

$$R^{(2)\beta}_{\mu\alpha\nu} \equiv -h^{\beta}_{\gamma} R^{(1)\gamma}_{\mu\alpha\nu} + A^{(2)\beta}_{\mu\alpha\nu} + Q^{(2)\beta}_{\mu\alpha\nu} + G^{(2)\beta}_{\mu\alpha\nu} + W^{(2)\beta}_{\mu\alpha\nu} + 4V^{(2)\beta}_{\mu\alpha\nu} \tag{25}$$

where we have used the notations:

$$A^{(2)\beta}_{\mu\alpha\nu} \equiv 2\left(h_{st} h^{st} R^{(0)\beta}_{\mu\alpha\nu} - h^{\beta s} h_{rs} R^{(0)r}_{\mu\alpha\nu}\right), \tag{26}$$

$$Q^{(2)\beta}_{\mu\alpha\nu} \equiv g^{(0)\beta q} g^{(0)}_{\rho\gamma}\left(H^{(1)\gamma}_{\mu\nu} H^{(1)\rho}_{q\alpha} - H^{(1)\gamma}_{\mu\alpha} H^{(1)\rho}_{q\nu}\right), \tag{27}$$

$$G^{(2)\beta}_{\mu\alpha\nu} \equiv h^{\beta s} h_{\rho s}\left(\Gamma^{(0)\rho}_{\mu\nu,\alpha} - \Gamma^{(0)\rho}_{\mu\alpha,\nu}\right) + h^{st} h_{st}\left(\Gamma^{(0)\beta}_{\mu\alpha,\nu} - \Gamma^{(0)\beta}_{\mu\nu,\alpha}\right), \tag{28}$$

$$W^{(2)\beta}_{\mu\alpha\nu} \equiv \sum_{i=1}^{4} W^{(2)\beta}_{i\mu\alpha\nu} \tag{29}$$

and $V^{(2)\beta}_{\mu\alpha\nu}$ and $W^{(2)\beta}_{i\mu\alpha\nu}$ (i=1,...4) are given by expressions (20) - (24) respectively.

Discussion

The main result of this paper is that the second-order Riemann tensor and therefore the gravitational Lagrangian can be expressed only through the background variables $g^{(0)\mu\nu}, \Gamma^{(0)\alpha}_{\mu\nu}, \Gamma^{(0)\alpha}_{\mu\nu,\delta}$ and the fluctuation variables $h_{\mu\nu}, H^{(1)\alpha}_{\mu\nu}, H^{(1)\alpha}_{\mu\nu|\delta}$. Note that to each background variable corresponds a fluctuation variable. There is also a difference - for example to the background *non-tensor* connection $\Gamma^{(1)\alpha}_{\mu\nu}$ corresponds the first fluctuation *tensor* connection $H^{(1)\alpha}_{\mu\nu}$; to the *partial* derivative $\Gamma^{(0)\alpha}_{\mu\nu,\delta}$ corresponds the *covariant* derivative $H^{(1)\alpha}_{\mu\nu|\delta}$.

It is seen also that the structure of the second-order Riemann tensor (25) is rather complicated, unlike the simplified expression, obtained in [2]. The terms $R^{(1)\beta}_{\mu\alpha\nu} \equiv H^{(1)\beta}_{\mu\nu|\alpha} - H^{(1)\beta}_{\mu\alpha,\nu}$ and also the quadratic connection term $\left(H^{(1)\gamma}_{\mu\nu} H^{(1)\rho}_{\gamma\alpha} - H^{(1)\gamma}_{\mu\alpha} H^{(1)\rho}_{\gamma\nu}\right)$ are present in [2] and also in this paper. The sum of these terms, although comprised of third-rank tensors and their covariant derivatives, is very similar to the expression for the field strength $G^{\alpha}_{\mu\nu}$ in non - abelian gauge theories [4, 5]:

$$G^{a}_{\mu\nu} \equiv \partial_{\mu} A^{a}_{\nu} - \partial_{\nu} A_{\mu} + g e^{abc} A^{b}_{\mu} A^{c}_{\nu}; \qquad g = \text{const} \tag{30}$$

where $A^{a}_{\mu\nu}$ is a vector non-abelian gauge field. Having in mind the well-known analogy between the theory of gravity and the non-abelian gauge field theory, one may assign to $A^{a}_{\mu\nu}$ the Christoffel connection symbols of second kind $\Gamma^{a}_{\mu\nu}$, which play a similar to $A^{a}_{\mu\nu}$ role, but in general relativity. The investigation on this subject is far from being completed and that is why some of the subsequent papers will be devoted to the some aspects of this theory (particularly the integral formulation).

'It is important to mention that besides the familiar from [2] terms, new terms $V^{\beta}_{\mu\alpha\nu}, W^{\beta}_{\mu\alpha\nu}, G^{\beta}_{\mu\alpha\nu}$ have appeared in the second-order Riemann tensor (25) in this paper. The sum of the terms $V^{(2)\beta}_{\mu\alpha\nu}$ (20) and $W^{(2)\beta}_{i\,\mu\alpha\nu}$ (i = 1, 2, 3, 4)(21) - (24) will be called a 'potential' term of interacting background and fluctuation connections.

Another interesting consequence from the performed calculations is the possibility to define new canonical variables. The fact that the fluctuation metric tensor $h_{\mu\nu}$ enters the Lagrangian without any partial or covariant derivatives (remember they have been eliminated by use of (13)) means that a conjugate canonical impulse cannot be defined similarly to the way this is done in canonical gravity theory [6] through the relation:

$$(31) \qquad p^{\mu\nu} \equiv \frac{\partial L}{\partial \dot{g}} \text{ (the dot means a time derivative).}$$

However, a generalized canonical impulse.

$$(32) \qquad \Pi_{\alpha}^{\mu\nu\delta} \equiv \frac{\partial L}{\partial H^{\alpha}_{\mu\nu|\delta}}$$

similar to the one, defined in [2] by the formulae:

$$(33) \qquad \pi^{\mu\nu\rho} \equiv \frac{\partial L}{\partial h_{\mu\nu|\rho}}$$

can be introduced. Also, it can be noted that expression (28) for $G^{(2)\beta}_{\mu\alpha\nu}$ contains partial derivatives of the background connection $\Gamma^{(0)\alpha}_{\mu\nu}$, which makes it possible to define canonical impulses:

$$(34) \qquad \Pi_{\Gamma} \equiv \frac{\partial L}{\partial \Gamma^{(0)\alpha}_{\mu\nu}} .$$

It can be checked that while $\Gamma^{(0)}_{\mu\nu}$ is not a tensor, for the case of the Lagrangian (I.24) - (I.27) Π_{Γ} will be a tensor quantity. The question about canonical variables will be treated in a subsequent publication.

Acknowledgements The author is grateful to Dr S. Manoff from the Theoretical Physics Department at the Institute for Nuclear Research and Nuclear Energy in the Bulgarian Academy of Sciences for the numerous and helpful discussions, advises and critical remarks.

References

1. D i m i t r o v , B.G. - Preceeding paper I.
2. B a r n e b y, T. A. - Phys. Rev. D, 10, 1974, 1741 - 1748.
3. M i s h t e n k o, A.S., A.T. F o m e n k o. A Course of Differential Geometry and Topology. Moscow, Moscow University Publishing House.
4. C h e n g, T a-P e i, L i n g F o n g L i. Gauge Theory of Elementary Particle Physics. Oxford, Claredon Press, 1984.

5. Ryder, Lewis H. Quantum Field Theory. Cambridge, Cambridge University Press, 1985.
6. Barvinsky, A. O. - Physics Reports, 230, 1993, p. 237.

Received 15. XI. 1995

Върху теорията на пертурбативната гравитация и квантовата гравитация на изкривен фон.
II. Приложение на ковариантни производни от трети ранг

Богдан Димитров

(Резюме)

В тази втора работа са опростени получените в предишната работа изрази за гравитационния Лагражиан от първи и от втори порядък. Въведени са ковариантни производни на тензори от втори и трети порядък, чрез които гравитационният Лагражиан придобива сравнително компактен вид. Разкрит е също физическият смисъл на тензорните членове в Лагражиана.